J Math Chem (2012) 50:2659–2670
DOI 10.1007/s10910-012-0055-1

ORIGINAL PAPER

Tridiagonal treatment for the Schrödinger equation with a noncentral electric dipole ring-shaped potential

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Received: 5 June 2012 / Accepted: 12 July 2012 / Published online: 27 July 2012

Abstract A noncentral potential is proposed in which the noncentral electric dipole and a ring-shaped component $\cos^2\theta/r^2\sin^2\theta$ are included. The exactly complete solutions of the Schrödinger equation with this potential is investigated by working in a complete square integrable basis that supports an infinite tridiagonal matrix representation of the wave operator. The solutions obtained are for all energies, the discrete (for bound states) as well as the continuous (for scattering states). The expansion coefficients of the wavefunctions (both angular and radial) are written in terms of orthogonal polynomials satisfying three-term recursion relation. The discrete spectrum of the bound states is obtained by diagonalization of the radial recursion relation.

Keywords Noncentral electric dipole potential · Tridiagonal matrix representation · Square integrable basis · Orthogonal polynomial

1 Introduction

It is well known that the exact solutions of wave equation with some central physical potentials play an important role in quantum mechanics, which provided strong evidence supporting the correctness of the quantum theory. For example, the exact solutions of the Schrödinger equation for a particle in the Coulomb potential or the harmonic oscillator in three dimensions [1] were an important milestone at the beginning stage of quantum mechanics, the concepts of the Coulomb potential or the harmonic oscillator give us a good first approximation for understanding the spectroscopy and the structure of diatomic molecules in their ground electronic states.

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In recent years, the noncentral potentials have been in the focus of some studies in various fields since the advance of the Coulomb ring-shaped potential and the oscillatory ring-shaped potential [2–12]. This partly due to the occurrence of 'accidental' degeneracy and 'hidden' symmetry in those noncentral potentials. Moreover, the possible applications in quantum chemistry and nuclear physics, such as the use of these noncentral potentials in the descriptions of ring-shaped molecules like benzene and the interactions between deformed pair of nuclei. New methods were introduced and developed for solving the wave equation with this kind of noncentral potential. The Kustaanheimo–Stiefel (KS) transformation [11,12] the Darboux transformation [13], the supersymmetric quantum mechanics and shape invariance method [14,15], and the Nikiforov–Uvarov (NU) method [16–18] are few among the many methods.

The interaction of a charged particle with the noncentral electric dipole potential, $V(r,\theta) = \cos\theta/r^2$(in spherical coordinates), is a fundamental problem, which received a lot of attention since the early days of nuclear and molecular physics [19,20]. In the latter, this interest was fueled by the observation that electron capture by a frozen molecule with a permanent electric dipole moment could take place only if the dipole moment exceeds a certain minimum critical value [21–24]. It should be noted that besides Alhaidari's pioneering contribution [25], no noncentral electric dipole potential appearing in the literature mentioned above. The major reason behind this fact may be that the noncentral electric dipole potential was believed not to belong to any of the established classes of exactly solvable potentials until Alhaidari, using the tridiagonalization program, made it a new member of the exactly solvable potentials [26,27].

In general, the noncentral ring-shaped potential has the form

$$V(\vec{r}) = V_r(r) + r^{-2}V_\theta(\theta). \tag{1}$$

Where

$$V_r(r) = \frac{A}{r} + \frac{B}{r^2} + Cr^2, \quad V_\theta(\theta) = \eta\cos\theta + f(\theta). \tag{2}$$

Then one could treat the special cases corresponding to specific choices for the set of potential parameters $\{A, B, C, \eta\}$ and the angle-dependent $f(\theta)$. In our recent study [28], $f(\theta)$ is selected as the novel angle-dependent (NAD) potential

$$f(\theta) = \frac{\hbar^2}{2\mu}\left(\frac{\gamma + \beta\sin^2\theta + \alpha\sin^4\theta}{\sin^2\theta\cos^2\theta}\right), \tag{3}$$

which is suggested for the first time by Berkdemir and the radial part $V_r(r)$ is selected as the Coulomb potential or the harmonic oscillator potential, respectively [29]. Exact solutions are obtained in the Schrödinger picture by means of the Nikiforov–Uvarov (N-U) method. In present work, if we select $B = C = 0$, and $f(\theta) = \cos^2\theta/\sin^2\theta$, another noncentral electric dipole ring-shaped potential is given as follows

$$V(r,\theta) = \frac{Q}{r} + \frac{\sigma\cos\theta}{r^2} + \frac{q\cos^2\theta}{r^2\sin^2\theta}. \tag{4}$$

Where σ is the electric dipole moment, Q its net charge and the parameter q is related to the property of the ring-shaped term. Therefore, the physics will be different from that mentioned above in some aspects. In this work, motivated by Alhaidari's contribution, we attempt to investigate the exact solution of the Schrödinger equation with the potential (4) by using the tridiagonalization program.

The remainder of this paper is arranged as follows. In Sect. 2, the tridiagonalization program and the associated L^2 basis are outlined briefly. In Sect. 3, the Schrödinger equation with this quantum system is separated into the angular and radial components. In Sect. 4, the resulting three-term recursion relation for the expansion coefficients of the wavefunctions (both angular and radial) are realized by tridiagonalization techniques. The solution obtained is for all energies, the discrete (for bound states) as well as the continuous (for scattering states). The discrete spectrum for the bound states is obtained with the diagonalization of the radial recursion relation. Section 5 is devoted to some concluding remarks and discusses.

2 Basis for the L^2 function space

Recently, Alhaidari has introduced the tridiagonalization program for solving the wave equation with solvable potentials. The main objective and motivation of this program is to find solutions of new problems that could not be solved by the traditional methods (the diagonal program). Examples are the electric quadrupole potential in two dimensions [30] the hyperbolic single wave potential in one dimension [31] and the Yukawa potential [32], etc. As expected, of course, the tridiagonal program does give the traditional solutions automatically [33].

In this program, one does not require a diagonal representation of the eigenvalue wave operator. One only require the matrix representation of wave operator be tridiagonal and symmetric. That is, the action of the wave operator on the elements of the basis is allowed to take the general form $(H-E)\,|\phi_m\rangle \approx |\phi_m\rangle + |\phi_{m-1}\rangle + |\phi_{m+1}\rangle$ and such that

$$\langle\phi_n\,|H-E|\,\phi_m\rangle = (a_n - z)\,\delta_{n,m} + b_n\delta_{n,m-1} + b_{n-1}\delta_{n,m+1}. \tag{5}$$

Where z and the coefficients $\{a_n, b_n\}_{n=0}^{\infty}$ are real and, in general, functions of the energy, angular momentum, and potential parameters. Therefore, the matrix wave equation, which is obtained by expanding $|\psi\rangle$ in $(H-E)\,|\psi\rangle = 0$ as $\Sigma_m f_m\,|\phi_m\rangle$ and projecting on the left by $\langle\phi_n|$, resulting in the following three-term recursion relation

$$z f_n = a_n f_n + b_{n-1} f_{n-1} + b_n f_{n+1}. \tag{6}$$

Consequently, the problem of solving the wave equation is translated into finding solutions of the recursion relation for the expansion coefficients of the wavefunction ψ. In most but not all cases, this recursion relation is solved easily by correspondence with those for well-known orthogonal polynomials. Moreover, the representation Eq. (5)

clearly shows that the discrete spectrum is easily obtained by diagonalization which requires that

$$b_n = 0 \quad \text{and} \quad a_n - z = 0, \tag{7}$$

for all n.

In the configuration space, with coordinate x, the wavefunction $\psi_E(x)$ is expanded as $\sum_{n=0}^{\infty} f_n(E)\phi_n(x)$, where L^2 basis functions $\phi_n(x)$ could generally be written as

$$\phi_n(x) = A_n w_n(x) P_n(x). \tag{8}$$

A_n is a normalization constant, $P_n(x)$ is a polynomial of degree n in x, and $w_n(x)$ is the weight function satisfies $w_n(x_\pm) = 0$, where $x_-(x_+)$ is the left (right) boundary of configuration space. In general, two kinds of spaces are useful, one is where $x_\pm$ are finite and

$$W_\pm(x) = (x - x_-)^\alpha (x - x_+)^\beta, \quad P_n(x) = {}_2F_1(-n, b, c; x). \tag{9}$$

The other is semi-infinite, where x_- is finite, x_+ is infinite, and has the proper characteristic

$$w_n(x_\pm) = (x - x_-)^\alpha e^{-\beta(x-x_-)}, \quad P_n(x) = {}_1F_1(-n, c; x). \tag{10}$$

Where ${}_2F_1\,(-n, b, c; x)$ is the hypergeometric function and ${}_1F_1\,(-n, c; x)$ is the confluent hypergeometric function. The parameters α, β, b and c are real with α and β positive. They are related to the physical parameters of corresponding problem and may also depend (for bound states) on the index n.

3 Noncentral separable potentials in spherical coordinates

Throughout this work the atomic units ($\hbar = M = e = 1$) are employed for simplicity. The three-dimensional time-independent Schrödinger equation in a potential $V(\vec{r})$ reads

$$\left[-\frac{1}{2}\vec{\nabla}^2 + V(\vec{r}) - E\right]|\psi(\vec{r})\rangle = 0, \tag{11}$$

where the energy E is either discrete or continuous, and $\vec{\nabla}$ is the three-dimensional Laplacian. In spherical coordinates, this wave equation is separable for potentials of the form

$$V(\vec{r}) = V_r + \frac{1}{r^2}\left[V_\theta(x) + \frac{1}{1-x^2}V_\varphi(\varphi)\right], \tag{12}$$

where $x = \cos\theta$. Thus, if we write the wavefunction as $\psi(r,\theta,\varphi) = r^{-1}R(r)\Theta(\theta)\Phi(\varphi)$, then the Eq. (11) with potential (12) gets separated as follows:

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}\varphi^2} - 2V_\varphi + 2E_\varphi\right)\Phi(\varphi) = 0, \tag{13}$$

$$\left[-\frac{1}{2}\left(1-x^2\right)\frac{\mathrm{d}^2}{\mathrm{d}x^2} + x\frac{\mathrm{d}}{\mathrm{d}x} + \frac{E_\varphi}{1-x^2} + V_\theta(x) - E_\theta\right]\Theta(x) = 0, \tag{14}$$

$$\left[-\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{E_\theta}{r^2} + V_r - E\right]R(r) = 0. \tag{15}$$

Where the separation constants E_φ and E_θ are real and dimensionless. In spherical coordinates, the square integrability of the L^2 basis is with respect to the following integration measures:

$$\int |\psi|^2 \mathrm{d}^3\vec{r} = \int_0^\infty |R|^2\,\mathrm{d}r \int_{-1}^{1} |\Theta|^2\,\mathrm{d}x \int_0^{2\pi} |\Phi|^2\,\mathrm{d}\varphi. \tag{16}$$

The components of the wavefunction are also required to satisfy the boundary conditions: $R(0) = R(\infty) = 0$, $\Phi(\varphi) = \Phi(\varphi + 2\pi)$ and $\Theta(0)$ and $\Theta(\pi)$ are of finite. If the potential $V(\vec{r})$ is specialized to the case $V_\varphi = 0$, the normalized solution of Eq. (13) that satisfies the boundary conditions is

$$\Phi_m(\varphi) = \frac{1}{\sqrt{2\pi}}\exp(im\varphi), \quad m = 0, \pm1, \pm2\ldots. \tag{17}$$

giving $E_\varphi = \frac{1}{2}m^2$.

4 The exact solutions of the Schrödinger equation

First, let us study the θ-dependent Eq. (14), where

$$V_\theta(x) = \sigma x + \frac{qx^2}{1-x^2}. \tag{18}$$

We can expand the angular wavefunction, $\Theta(\theta)$, in a complete square integrable basis functions $\{\chi_n(x)\}_{n=0}^\infty$ as $\Theta(\theta) = \Sigma_{n=0}^\infty f_n^m(E_\theta)\chi_n(x)$. These basis elements must satisfy the boundary conditions in the configuration space with coordinate $x \in [-1,+1]$. They are written as

$$\chi_n(x) = A_n(1-x)^\alpha(1+x)^\beta P_n^{(\mu,\nu)}(x). \tag{19}$$

Where $P_n^{(\mu,\nu)}(x)$ is the Jacobi polynomial of order n and $n = 0, 1, 2, \ldots$. The dimensionless real parameters $\alpha, \beta \geq 0$, $\mu, \nu > -1$ and A_n is the normalization constant associated with the orthogonality of the Jacobi polynomials,

$$A_n = \sqrt{\frac{(2n+\mu+\nu+1)\,\Gamma(n+1)\Gamma(n+\mu+\nu+1)}{2^{\mu+\nu+1}\Gamma(n+\mu+1)\Gamma(n+\nu+1)}}. \tag{20}$$

If we take the θ-angular Hamiltonian

$$H_\theta = -\frac{1}{2}\left(1-x^2\right)\frac{\mathrm{d}^2}{\mathrm{d}x^2} + x\frac{\mathrm{d}}{\mathrm{d}x} + \frac{qx^2+E_\varphi}{1-x^2} + \sigma x, \tag{21}$$

and using the following differential formulas of the Jacobi polynomials

$$\left\{\left(1-x^2\right)\frac{\mathrm{d}^2}{\mathrm{d}x^2} - [(\mu+\nu+2)\,x+\mu-\nu]\frac{\mathrm{d}}{\mathrm{d}x} + n(n+\mu+\nu+1)\right\} P_n^{(\mu,\nu)} = 0, \tag{22}$$

$$\left(1-x^2\right)\frac{\mathrm{d}}{\mathrm{d}x}P_n^{(\mu,\nu)} = -n\left(x+\frac{\nu-\mu}{2n+\mu+\nu}\right)P_n^{(\mu,\nu)} + 2\frac{(n+\mu)\,(n+\nu)}{2n+\mu+\nu}P_{n-1}^{(\mu,\nu)} = 0, \tag{23}$$

the action of the θ-angular wave operator $(H_\theta - E_\theta)$ on the basis element (19) is obtained as

$$\begin{aligned}|H_\theta - E_\theta|\,\chi_n\rangle = &\left[\frac{n}{2}\left(x+\frac{\nu-\mu}{2n+\mu+\nu}\right)\left(\frac{\mu-2\alpha}{1-x}+\frac{2\beta-\nu}{1+x}\right) - \frac{\alpha^2}{2}\left(\frac{1+x}{1-x}\right) - \frac{\beta^2}{2}\left(\frac{1-x}{1+x}\right)\right.\\ &\left.+\left(\alpha\beta+\frac{\alpha+\beta}{2}\right)+\frac{n}{2}(n+\mu+\nu+1)+\sigma x+\frac{qx^2+m^2/2}{\left(1-x^2\right)}-E_\theta\right]|\chi_n\rangle\\ &-\frac{(n+\mu)\,(n+\nu)}{2n+\mu+\nu}\left(\frac{\mu-2\alpha}{1-x}+\frac{2\beta-\nu}{1+x}\right)\frac{A_n}{A_{n-1}}\,|\chi_{n-1}\rangle.\end{aligned} \tag{24}$$

Considering the following recurrence relation and the orthogonality for the Jacobi polynomials

$$\begin{aligned}x\,P_n^{(\mu,\nu)} = &\frac{\nu^2-\mu^2}{(2n+\mu+\nu)\,(2n+\mu+\nu+2)}P_n^{(\mu,\nu)} + \frac{2\,(n+\mu)\,(n+\nu)}{(2n+\mu+\nu)\,(2n+\mu+\nu+1)}P_{n-1}^{(\mu,\nu)}\\ &+\frac{2\,(n+1)\,(n+\mu+\nu+1)}{(2n+\mu+\nu+1)\,(2n+\mu+\nu+2)}P_{n+1}^{(\mu,\nu)},\end{aligned} \tag{25}$$

$$\int_{-1}^{1}(1-x)^\mu\,(1+x)^\nu\,P_n^{(\mu,\nu)}P_{n'}^{(\mu,\nu)}\mathrm{d}x = \frac{2^{\mu+\nu+1}\Gamma(n+\mu+1)\Gamma(n+\nu+1)}{(2n+\mu+\nu+1)\Gamma(n+1)\Gamma(n+\mu+\nu+1)}\delta_{nn'}. \tag{26}$$

From which, a tridiagonal matrix representation for the θ-angular wave operator, $\langle \chi_n | H_\theta - E_\theta | \chi_{n'} \rangle$, is achievable with choices of the parameters $\alpha^2 = \beta^2, \alpha = \mu/2, \beta = \nu/2$, and $\mu^2 + \nu^2 = 2\left(m^2 + 2q\right)$. Thus, $\mu^2 = \nu^2 = m^2 + 2q$ requiring that the azimuthal quantum number m satisfy $m^2 > -2q$.To this end, we project the action of the angular wave operator given by Eq. (24) on the basis elements from left, a tridiagonal structure for angular wave operator is obtained as follows:

$$2\langle \chi_n | H_\theta - E_\theta | \chi_{n'} \rangle = \left[\left(n+\mu+\frac{1}{2}\right)^2 - \left(\gamma+\frac{1}{2}\right)^2 - 2q\right]\delta_{n,n'} + \sigma\sqrt{\frac{n\,(n+2\mu)}{(n+\mu)^2-1/4}}\delta_{n,n'-1} + \sigma\sqrt{\frac{(n+1)\,(n+2\mu+1)}{(n+\mu+1)^2-1/4}}\delta_{n,n'+1}. \quad (27)$$

Where $\mu = \sqrt{m^2+2q}$. We have also introduced the dimensionless parameter γ by writing $2E_\theta = \gamma(\gamma+1) = (\gamma+1/2)^2 - 1/4$. Equation (15) shows that γ plays the role of the angular momentum quantum number ℓ in spherically symmetric problems. However, unlike ℓ that assumes non-negative integral values, γ is a continuous parameter that could be positive or negative. For positive parameter E_θ, γ must be either greater than zero or less than -1. The reader can refer to [26] for more information in support of this argument. The tridiagonal matrix representation of the θ-angular wave operator in (27) makes the θ-angular Eq. (14) equivalent to the following three-term recursion relation for the expansion coefficients of the θ-angular component of the wavefunction

$$\left(\gamma+\frac{1}{2}\right)^2 f_n^m = \left[\left(n+\mu+\frac{1}{2}\right)^2 - 2q\right] f_n^m + \sigma\sqrt{\frac{n\,(n+2\mu)}{(n+\mu)^2-1/4}} f_{n-1}^m + \sigma\sqrt{\frac{(n+1)\,(n+2\mu+1)}{(n+\mu+1)^2-1/4}} f_{n+1}^m. \quad (28)$$

The resulting three-term recursion relation (6) could be written in terms of the polynomial defined by

$$f_n^m(E_\theta) = \frac{\sqrt{(n+\mu+1/2)\,\Gamma(n+1)\Gamma(n+\mu+1)}}{2^\mu\,\Gamma(n+\mu+1)} P_n^m(\gamma,\sigma). \quad (29)$$

In which case Eq. (28) reads

$$\left(\gamma+\frac{1}{2}\right)^2 p_n^m = \left[\left(n+\mu+\frac{1}{2}\right)^2 - 2q\right] P_n^m + \sigma\frac{(n+\mu)}{(n+\mu+1/2)} P_{n-1}^m + \sigma\frac{(n+1)\,(n+2\mu+1)}{(n+\mu+1)\,(n+\mu+1/2)} P_{n+1}^m. \quad (30)$$

The θ-angular component of the wavefunction (with arbitrary normalization by taking $P_0^m(\gamma,\sigma)=1$) could be written in terms of the L^2 basis as

$$\Theta(\theta)=\Sigma_n P_n^m(\gamma,\sigma)\chi_n(\theta). \tag{31}$$

We are now in the position to carry out the solutions of the radial Eq. (15). We can expand the radial wavefunction, $R(r)$, in a complete square integrable basis functions $\{\xi_k(y)\}_{k=0}^{\infty}$ as $R(r)=\Sigma_{k=0}^{\infty}g_k(E)\xi_k(y)$. These basis elements are compatible with the domain of the radial Hamiltonian H_r and must satisfy the boundary conditions in the configuration space with coordinate $y\in[0,\infty]$. They are written as

$$\xi_k(y)=B_k y^{\tau}e^{-y/2}L_k^{\delta}(y), \tag{32}$$

where $y=\lambda r, k=0,1,2\ldots$, and $L_k^{\delta}(y)$ is the Laguerre polynomial of order k. The real parameter λ is positive and carries the dimension of inverse length. The dimensionless parameters $\tau>0$ and $\delta>-1$, and the normalization constant $B_k=\sqrt{\lambda\Gamma(k+1)/\Gamma(k+\delta+1)}$. We take the radial Hamiltonian as

$$H_r=-\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}r^2}+\frac{Q}{r}+\frac{E_\theta}{r^2}. \tag{33}$$

Using the following differential formulas for the Laguerre polynomial

$$\left[y\frac{\mathrm{d}^2}{\mathrm{d}y^2}+(\sigma+1-y)\frac{\mathrm{d}}{\mathrm{d}y}+k\right]L_k^{\delta}(y)=0, \tag{34}$$

$$y\frac{\mathrm{d}}{\mathrm{d}y}L_k^{\delta}(y)=kL_k^{\delta}(y)-(k+\delta)\,L_{k-1}^{\delta}(y), \tag{35}$$

we obtain

$$\begin{aligned}&|H_r-E|\,\xi_k\rangle\\&=\frac{\lambda^2}{2}\left[\frac{k}{y}\left(1+\frac{\delta+1-2\tau}{y}\right)+\frac{\gamma(\gamma+1)-\tau(\tau-1)}{y^2}+\frac{\tau}{y}-\frac{1}{4}+\frac{2Q}{\lambda y}-\frac{2E}{\lambda^2}\right]|\xi_k\rangle\\&\quad+\frac{\lambda^2}{2}\frac{(k+\delta)(2\tau-\delta-1)}{y^2}\frac{B_k}{B_{k-1}}\,|\xi_{k-1}\rangle.\end{aligned} \tag{36}$$

Let us recall the following recurrence relation and the orthogonality for the Laguerre polynomials

$$yL_k^{\delta}(y)=(2k+\delta+1)\,L_k^{\delta}(y)-(k+\delta)\,L_{k-1}^{\delta}(y)+(k+1)\,L_{k+1}^{\delta}(y), \tag{37}$$

$$\int_0^{\infty}y^{\delta}e^{-y}L_k^{\delta}(y)L_{k'}^{\delta}(y)\mathrm{d}y=\frac{\Gamma(k+\delta+1)}{\Gamma(k+1)}\delta_{kk'}. \tag{38}$$

Thus, a tridiagonal matrix representation for the radial wave operator $\langle \xi_k \mid H_r - E_r \mid \xi_{k'} \rangle$ is achievable if and only if $\delta = 2\tau - 1$ and $\gamma\,(\gamma+1) = \tau(\tau-1)$, that is

$$\tau = \begin{cases} \gamma + 1, & \gamma \geq 0 \\ -\gamma, & \gamma \leq -1 \end{cases} \tag{39}$$

Which makes τ always greater than or equal to $+1$ and $\delta = \pm\,(2\gamma+1)$ for $\pm\gamma > 0$. We obtain the following tridiagonal matrix representation of the radial wave operator

$$\frac{2}{\lambda^2}\,\langle \xi_k \mid H_r - E \mid \xi_{k'} \rangle = \left[\left(\frac{1}{4} - \frac{2E}{\lambda^2}\right)(2k+\delta+1) + \frac{2Q}{\lambda}\right]\delta_{k,k'} + \left(\frac{1}{4} + \frac{2E}{\lambda^2}\right)\sqrt{k\,(k+\delta)}\,\delta_{k,k'+1} + \left(\frac{1}{4} + \frac{2E}{\lambda^2}\right)\sqrt{(k+1)\,(k+\delta+1)}\,\delta_{k,k'-1}. \tag{40}$$

Therefore, the resulting three-term recursion relation for the expansion coefficients of the radial wavefunction becomes

$$2\left[\left(k+\frac{\delta+1}{2}\right)\frac{\eta_-}{\eta_+} - \frac{Q/\lambda}{\eta_+}\right] g_k - \sqrt{k\,(k+\delta)}\,g_{k-1} - \sqrt{(k+1)\,(k+\delta+1)}\,g_{k+1} = 0, \tag{41}$$

where $\eta_\pm = \frac{2E}{\lambda^2} \pm \frac{1}{4}$.

It is shown in Eq. (4) that this new noncentral potential is independent of angle φ, the solutions of this quantum system must be symmetric about the z-axis. Rewriting the recursion relation (41) in terms of the polynomials defined by

$$\tilde{P}_k(Q;\,E) = \sqrt{\Gamma(k+2\tau)/\Gamma(k+1)}\,g_k(E), \tag{42}$$

we obtain a more familiar recursion relation as follows:

$$2\left[\left(k + \frac{\delta+1}{2} + \frac{2Q}{\lambda}\right)\cos\varphi - \frac{2Q}{\lambda}\right]\tilde{P}_k - (k+\delta)\,\tilde{P}_{k-1} - (k+1)\,\tilde{P}_{k+1} = 0, \tag{43}$$

where $\cos\varphi = \eta_-/\eta_+ = \frac{8E-\lambda^2}{8E+\lambda^2}$. Compare this three-term recursion relation to that of a special case of the Pollaczek polynomials, $P_k^{\mu}(\nu;\,z)$: [26,34,35]

$$2\,[(k+\mu+\nu)\,z - \nu]\,P_k^{\mu} = (k+2\mu-1)\,P_{k-1}^{\mu} + (k+1)\,P_{k-1}^{\mu}, \tag{44}$$

where $\mu > 0$, ν is real, $z = \cos\varphi$. It should be understood that the parameter z is not same as that appear in the Eq. (6) and $0 < \varphi < \pi$. The orthogonality relation of the Pollaczek polynomials is

$$\int_0^\infty \rho^\mu(\nu;z)P_k^\mu(\nu;z)P_{k'}^\mu(\nu;z)\mathrm{d}z = \frac{\Gamma(k+2\mu)}{(k+\mu+\nu)\Gamma(k+1)}\delta_{kk'}. \tag{45}$$

Where $\rho^\mu(\nu;z)$ is the weight function. Therefore, we can write $\tilde{P}_k(Q;E) \propto P_k^{\frac{\delta+1}{2}}\left(\frac{2Q}{\lambda};\cos\varphi\right)$. However, this is valid only within the permissible range of values of the parameters. This means that the solution obtained as such is valid only for $E \geq 0$, i.e., for the continuum scattering states. Thus for the continuum case, we obtain the solution of the radial wavefunction in the tridiagonal representation as

$$R(r) = B_\gamma^E(\lambda r)^{\frac{\delta+1}{2}}e^{-\lambda r/2}\Sigma_k\sum_{k=0}^{\infty}P_k^{\frac{\delta+1}{2}}\left(\frac{2Q}{\lambda},\cos\varphi\right)L_k^\delta(\lambda r). \tag{46}$$

The normalization constant B_γ^E depends on γ and the energy but, otherwise, independent k [26].

To obtain the discrete energy representation, we should impose the diagonalization constraint on the tridiagonal matrix representation (40), the first condition $b_n = 0$, giving $\lambda^2 = -8E$. Since λ must be positive, this requires that energy must be negative. The second condition $a_n - z = 0$, gives the following

$$\lambda_k = \frac{-2Q}{(k+\tau)} = \begin{cases} -2Q/(k+\gamma+1), & \gamma \geq 0 \\ -2Q/(k-\gamma), & \gamma \leq -1 \end{cases} \tag{47}$$

Which requires that $Q < 0$. That is, the bound states exist only for the attractive Coulomb potential.Thus, the discrete bound states energy spectrum is

$$E_n = \frac{-Q^2}{2\,(k+\tau)^2} = \begin{cases} -Q^2/2(k+\gamma+1)^2, & \gamma \geq 0 \\ -Q^2/2(k-\gamma)^2, & \gamma \leq -1 \end{cases}. \tag{48}$$

This formula shows that the energy spectrum depends on the net charge Q and the angular parameter γ. Although angular parameter γ plays the role of the angular momentum quantum number in spherically symmetric problems, but γ is a continuous parameter that could be positive or negative. For positive parameter E_θ, γ must be either greater than zero or less than -1. The three-term recursion relation (30) gives the parameter γ in terms q and σ. Consequently, the energy spectrum depends implicitly on these physical parameters. The corresponding radial component of the discrete bound states wavefunctions is

$$R(r) = \sum_k P_k(Q;E)\xi_k(y). \tag{49}$$

Finally, the exact complete solutions of the Schrödinger equation with the model potential (4) is expressed as follows:

$$\psi(r,\theta,\varphi) = \frac{r^{-1}}{\sqrt{2\pi}} R(r)\Theta(\theta)\exp(\pm im\varphi), \quad m = 0, 1, 2 \ldots . \tag{50}$$

5 Conclusions

In this work, we have proposed a new noncentral electric dipole ring-shaped potential and obtained its exact solution by working in a complete square integrable basis that supports an infinite tridiagonal matrix representation of the wave operator. Exact solvability here means that we could obtain analytically the energy eigenvalues and the eigen-functions. Moreover, this solution is for all energies, the discrete (for bound states) as well as the continuous (for scattering states). The discrete spectrum of the bound states is obtained by diagonalization of the radial recursion relation. It is evident that the tridiagonalization program enables one to obtain exact solutions by mapping the wave equation into an equivalent three-term recursion relation for the expansion coefficients of the wavefunction. Consequently, finding solutions of the recursion relation is equivalent to solving the original wave equation. It is also pointed out that the exact results obtained in this work may have some potential applications in different fields. For example, it can be used to account for axial symmetry in system in physics and quantum chemistry.

Acknowledgments The author would like to thank the kind referees for their positive and invaluable suggestions which improved the manuscript greatly.

References

1. L.I. Schiff, *Quantum Mechanics,* 3rd edn. (McGraw-Hill, New York, 1955)
2. A.A. Makarov, J.A. Smorodinsky, K.H. Valiev, P. Winternitz, Nuovo Cimento A **52**, 1061 (1967)
3. A. Hautot, J. Math. Phys. **14**, 1320 (1973)
4. A. Khare, R.K. Bhaduri, Am. J. Phys. **62**, 1008 (1994)
5. M. Kibler, L.G. Mardoyan, G.S. Pogosyan, Int. J. Quant. Chem. **52**, 1301 (1994)
6. M. Kibler, G.H. Lamot, P. Winternitz, Int. J. Quant. Chem. **43**, 625 (1992)
7. H. Hartmann, Theor. Chim. Acta **24**, 201 (1972)
8. H. Hartmann, R. Schuck, J. Radtke, Theor. Chim. Acta **42**, 1 (1976)
9. H. Hartmann, D. Schuch, Int. J. Quant. Chem. **18**, 125 (1980)
10. C. Quesne, J. Phys. A: Math. Gen. **21**, 3093 (1988)
11. M. Kibler, T. Negadi, Int. J. Quant. Chem. **26**, 405 (1984)
12. M. Kibler, T. Negadi, Theor. Chim. Acta **66**, 31 (1984)
13. A. Schulze-Halberg, E. Zamora-Gallardo, J. J. Peña, Int. J. Quant. Chem. **109**, 1464 (2009)
14. F. Cooper, A. Khare, U. Sukhatme, Phys. Rep. **251**, 267 (1995)
15. R. Dutt, A. Khare, U.P. Sukhatme, Am. J. Phys. **56**, 163 (1988)
16. A.F. Nijkiforov, V.B. Uvarov, *Special Functions of Mathematical Physics* (Birkhauser, Basel, 1988)
17. F. Yaşuk, C. Berkdemir, A. Berkdemir, J. Phys. A: Math. Gen. **38**, 6579 (2005)
18. M. Aktaş, R. Sever, J. Math. Chem. **37**, 139 (2005)
19. E. Fermi, E. Teller, Phys. Rev. **72**, 399 (1947)
20. A.S. Wightman, Phys. Rev. **77**, 521 (1950)
21. K. Fox, J.E. Turner, J. Chem. Phys. **45**, 1142 (1966)
22. W.B. Brown, R.E. Roberts, J. Chem. Phys. **46**, 2006 (1967)
23. S.A. Coon, B.R. Holstein, Am. J. Phys. **70**, 513 (2002)
24. B. Jaramillo, H.N. Núñez-Yépez, A.L. Salas-Brito, Phys. Lett. A **374**, 2707 (2010)
25. A.D. Alhaidari, J. Phys. A: Math. Gen. **38**, 3409 (2005)

26. A.D. Alhaidari, Ann. Phys. **323**, 1709 (2008)
27. A.D. Alhaidari, H. Bahlouli, Phys. Rev. Lett. **100**, 110401 (2008)
28. G.Q. Huang-Fu, M.C. Zhang, J. Math. Chem. **50**, 1988 (2012)
29. V. Berkdemir, J. Math. Chem. **46**, 139 (2009)
30. A.D. Alhaidari, J. Phys. A: Math. Theor. **40**, 14843 (2007)
31. H. Bahlouli, A.D. Alhaidari, Phys. Scr. **81**, 025008 (2010)
32. H. Bahlouli, M.S. Abdelmonem, I.M. Nasser, Phys. Scr. **82**, 065005 (2010)
33. A.D. Alhaidari, Ann. Phys. **317**, 152 (2005)
34. T.S. Chihara, *An Introduction to Orthogonal Polynomial* (Gordon and Breach, New York, 1978)
35. M. Aunola, J. Phys. A: Math. Gen. **38**, 1279 (2005)